Yiddish word of the day

hilke pilke shpiler = הילקע פּילקע שפּילער

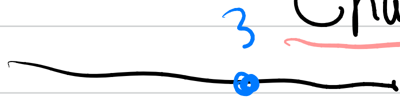
baseball player =

Yiddish expression/curse ect

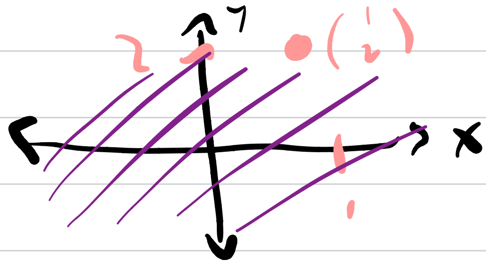
"zolst vaksn vi a tsibile, mit der kop in der erd" = זאָלסט וואַקסן ווי אַ ציבעלע, מיטן קאָפּ אין דער ערד!

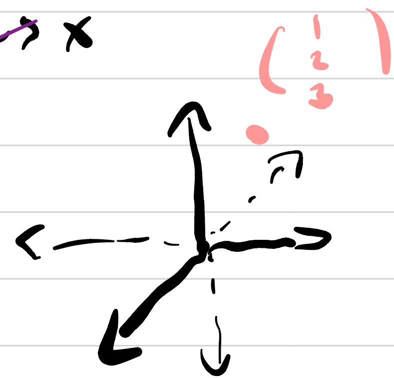
may you grow like an onion with your head in the ground. =

Chapter 2: $\mathbb{R}^n$

ex: $\mathbb{R}$ 

real line

$\mathbb{R}^2$: 

$\mathbb{R}^3$: 

Def: • $\mathbb{R}^n$ is the collection (set) of all n-tuples of real #s,

• an element in $\mathbb{R}^n$ is called an n-vector, $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

ex) $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ is a 4-vector in $\mathbb{R}^4$

ii) $\begin{pmatrix} -2 \\ -10 \end{pmatrix}$ is a 2-vector in $\mathbb{R}^2$

- Addition of two vectors in $\mathbb{R}^n$ is defined "component-wise"

$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ $\quad \vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ $\Rightarrow$ $\boxed{\vec{v}+\vec{w} = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{pmatrix}}$

- Multiplication by a "scalar" (ie a real #) is defined componentwise,
  - if c is a real # $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ is an n-vector, then

$$c\vec{v} = \begin{pmatrix} cv_1 \\ cv_2 \\ cv_3 \\ \vdots \\ cv_n \end{pmatrix}$$

Def: The standard basis for $\mathbb{R}^n$ are the following vectors.

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

ex: $\mathbb{R}^2$ the standard basis is

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\vec{e}_1 + 3\vec{e}_2$$

Def: Let $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$ be vectors in $\mathbb{R}^n$. Then a linear combination of these vectors is an expression like

$c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_m\vec{u}_m$ where $c_1, \dots, c_m$ are real #'s.

ex) Let $\vec{u}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$. Then $2\vec{u}_1 + 3\vec{u}_2 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 12 \\ 18 \end{pmatrix} = \begin{pmatrix} 16 \\ 24 \end{pmatrix}$ is a linear combination of $\vec{u}_1, \vec{u}_2$

ex: Is the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ a Linear combination of the vectors $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$?

If $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a LC of $\vec{u}_1$ and $\vec{u}_2$ then we have

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{for some \#'s } c_1, c_2$$

$$\begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -2c_2 \\ -c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - 2c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \begin{aligned} c_1 - 2c_2 &= 2 \\ c_1 - c_2 &= 3 \end{aligned}$$

$c_1 - 2c_2 = 2$ $\xrightarrow{R_2 \to R_2 - R_1}$ $c_1 - 2c_2 = 2$

$c_1 - c_2 = 3$ $\qquad\qquad\qquad\qquad c_2 = 1$

$c_1 - 2 = 2 \Rightarrow c_1 = 4$

Claim: $4\vec{u}_1 + \vec{u}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\parallel$

$\begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ✓

ex2: Is the vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ a LC of

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$?

Again, we want to see if

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ for some #'s c_1, c_2, c_3

$$\begin{pmatrix} c_1 \\ c_1 \\ -2c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ 3c_2 \\ 2c_2 \end{pmatrix} + \begin{pmatrix} c_3 \\ 0 \\ -c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - c_2 + c_3 \\ c_1 + 3c_2 + 0 \\ -2c_1 + 2c_2 - c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 3 & 0 & -2 \\ -2 & 2 & -1 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$c_3 = 3 \Rightarrow 4c_2 - 3 = -3 \Rightarrow 4c_2 = 0$ so $c_2 = 0$

$c_1 + 0 + 3 = 1 \Rightarrow c_1 = -2$

Check: $-2\begin{pmatrix}1\\1\\-2\end{pmatrix} + 3\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$ ✓

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ex: Is $\begin{pmatrix}1\\0\\1\end{pmatrix}$ a LC of $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}0\\2\\0\end{pmatrix}$

$$c_1\begin{pmatrix}1\\2\\1\end{pmatrix} + c_2\begin{pmatrix}0\\2\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} \longrightarrow \begin{pmatrix} c_1 + 0c_2 \\ 2c_1 + 2c_2 \\ c_1 + 0c_2 \end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{array}\right) \quad c_1 = 1 \quad 2(1) + 2c_2 = 0$$
$$c_2 = -1$$

<u>Thrm</u>: A vector $\vec{b}$ in $\mathbb{R}^n$ is a LC of the vectors $\vec{v}_1, \dots, \vec{v}_m$ if and only if

$$\left(\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m & \vec{b} \\ \downarrow & \downarrow & & \downarrow & \downarrow \end{array}\right)$$

is consistent = has a solution

We're saying that a vector $\vec{b}$ in $\mathbb{R}^n$ is LC of $\vec{v}_1, \dots, \vec{v}_m$

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the linear system of equations with matrix given by

has a solution

ex) Is the vector $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ a LC of $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -3 \\ -6 \\ -4 \end{pmatrix}$

$\vec{v}_3 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$

$$\Rightarrow \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 2 & -6 & 4 & 1 \\ 1 & -4 & 3 & -2 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_2 + R_3} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow \begin{array}{l} c_1 - 3 - 2 = 0 \Rightarrow c_1 = 5 \\ c_2 = 1 \\ c_3 = -1 \end{array}$$

Check: $\underbrace{5\vec{v}_1 + 1\vec{v}_2 - \vec{v}_3}$ = $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5-5 \\ 10-5-4 \\ 5-4-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \checkmark$$

Chapter 2.3 – Span

Def: Let $\vec{v}_1, \dots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. Then the span of $\vec{v}_1, \dots, \vec{v}_k$ is the collection (set) of all possible LC of vectors $\vec{v}_1, \dots, \vec{v}_k$

Denote this collection by $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$

Verb: If $\text{span}(\vec{v}_1, \ldots, \vec{v}_k) = \mathbb{R}^n$ then we say that the vectors $\vec{v}_1, \ldots, \vec{v}_k$ span $\mathbb{R}^n$

Recall: A vector $\vec{b}$ is a LC of vectors $\vec{v}_1, \ldots, \vec{v}_k$ is the same thing as

$$\left(\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k & \vec{b} \\ \downarrow & \downarrow & & \downarrow & \downarrow \end{array}\right)$$

this matrix having a solution.

ex) Is the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in span$\left(\overset{\vec{v}_1}{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}, \overset{\vec{v}_2}{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}, \overset{\vec{v}_3}{\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}} \right)$

- the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is in span(v_1, v_2, v_3)

if and only if

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 4 & 3 \end{array}\right)$$ has a solution.

We saw (see extra notes) that this matrix does not have a solution.

So, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is <u>NOT</u> in span(v_1, v_2, v_3)

ex) Is the vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$ in $\mathrm{span}\left(\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}2\\0\\4\end{pmatrix}\right)$?

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad v_1 \quad\quad v_2 \quad\quad v_3$

Again just have to see if $\begin{pmatrix}1\\2\\3\end{pmatrix}$ is LC of v_1, v_2, v_3

form matrix $\longrightarrow$ $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 4 & 3 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right)$

$c_3 = 1$

$c_2 = 1$

$c_1 + 2 = 1 \Rightarrow c_1 = -1$

Note: Only way A system is inconsistent is if we have a zero row = non-zero #

- So if $\vec{v}_1, \ldots, \vec{v}_k$ span $\mathbb{R}^n$ then for any vector $\vec{b}$ in $\mathbb{R}^n$ the linear system

$$\left(\begin{array}{ccc|c} v_1 & \cdots & \vec{v}_k & \vec{b} \\ \downarrow & & \downarrow & \downarrow \end{array}\right)$$

is consistent

$\Longrightarrow$ there won't be a zero row when it is put into echelon form.

$\Longleftrightarrow$ there is a pivot in every row

This implies that if $\vec{v}_1 \ldots \vec{v}_k$ span $\mathbb{R}^n$

then $\boxed{k \geq n}$

ex:

A line can't span $\mathbb{R}^2$

ex) Can $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1000 \\ 10 \\ -20 \\ 0 \end{pmatrix}$ span $\mathbb{R}^4$?

No, only 3 vectors.

Summary: Let $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_k \\ \downarrow & & \downarrow \end{pmatrix}_{n\times k}$ be a $n\times k$ matrix

Then the following are equivalent

1) For every vector $\vec{b}$ in $\mathbb{R}^n$ the matrix
$$\left(\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k & \vec{b} \\ \downarrow & \downarrow & & \downarrow & \downarrow \end{array}\right)$$
is consistent

2) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ span $\mathbb{R}^n$

3) There is a pivot in every row of A when A is in EF

ex) Use "language of spanning" to describe solution sets to equations.

Find the solution set to the equations

$$x_1 - 2x_2 + 3x_3 = 0 \qquad\qquad x_1 - 2x_2 + 3x_3 = 0$$
$$-x_1 + 4x_2 - 6x_3 = 0 \quad \Longrightarrow \quad 2x_2 - 3x_3 = 0$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right)$$

(Side Q: Do $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-2\\2\end{pmatrix}, \begin{pmatrix}3\\-3\end{pmatrix}$ span $\mathbb{R}^2$) Yes!

- $x_1 - 3x_3 + 3x_3 = 0 \Rightarrow x_1 = 0$
- $x_2 = \frac{3}{2}x_3$

Note, x_3 is a free variable.

$x_1 = 0$
$x_2 = 3/2\, x_3$
$x_3 = x_3$

$$\Longrightarrow \boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \text{Span}\begin{pmatrix} 0 \\ 3/2 \\ 1 \end{pmatrix}}$$

ex) Find the solution set to the homogeneous system with coefficient matrix $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 1 \\ 2 & 6 & 0 & 4 \\ 1 & -1 & 4 & -1 \end{pmatrix}$

has EF $\Rightarrow$ $\begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Note x_3, x_4 are free variables.

Call $x_3 = r$, $x_4 = s$ $\Rightarrow$ $x_1 + 3r - s = 0$, $x_2 - r + s = 0$ $\Rightarrow$ $x_1 = s - 3r$, $x_2 = -s + r$, $x_3 = r$, $x_4 = s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} s-3r \\ -s+r \\ r \\ s \end{pmatrix} = \begin{pmatrix} s \\ -s \\ 0 \\ s \end{pmatrix} + \begin{pmatrix} -3r \\ r \\ r \\ 0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + r \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}}$$

That is the solution set is all possible LC's of the vectors

$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ ie

the solution set is $\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right)}$